GLOBAL EXISTENCE AND BLOW-UP FOR A WEAKLY DISSIPATIVE μDP EQUATION

MARTIN KOHLMANN

Abstract. In this paper, we study a weakly dissipative variant of the periodic Degasperis-Procesi equation. We show the local well-posedness of the associated Cauchy problem in $H^s(\mathbb{S})$, $s > 3/2$, and discuss the precise blow-up scenario for $s = 3$. We also present explicit examples for globally existing solutions and blow-up.

1. Introduction

In recent years, the family

$$y_t = -(y_x u + b u_x y), \quad y = u - u_{xx}, \tag{1}$$

of nonlinear equations has been studied extensively; see [9, 12]. Here, $u(t,x)$ depends on a time variable $t \geq 0$ and a space variable x with $x \in \mathbb{S} \simeq \mathbb{R}/\mathbb{Z}$ for the periodic equation and $x \in \mathbb{R}$ in the non-periodic case. Equation (1) is called b-equation and (1) becomes the Camassa-Holm (CH) equation for $b = 2$ and the Degasperis-Procesi (DP) equation if $b = 3$. Moreover, the corresponding family of μ-equations, where

$$y = \mu(u) - u_{xx}, \quad \mu(u) = \int u(t,x)\,\mathrm{d}x$$

in (1), has been studied, e.g., in [13, 18]. It is known that the b-equation models the unidirectional motion of 1D water waves over a flat bed; for the hydrodynamical relevance we refer to, e.g., [2, 5] and [4, 6, 15]. It turned out that the b-equation is integrable only if $b \in \{2,3\}$ and in [18] the authors mention that a similar result is conjectured for the family of μ-equations. The Cauchy problems for the b-equation and its μ-variant have been discussed in [12, 18]. In particular, for $b \in \{2,3\}$, local well-posedness results in the periodic and in the real line-case as well as blow-up and criteria for the global existence of strong and weak solutions have been established; see, e.g., [3, 10, 23]. In addition, CH and DP and μCH and μDP admit peaked solitons, which make them attractive among the integrable equations; cf. [2, 5, 18].

In general, it is difficult to avoid energy dissipation mechanisms in the modeling of fluids. Ott and Sudan [20] discussed the KdV equation under the influence of energy dissipation. Ghidaglia [14] studied the behavior of solutions of the weakly dissipative KdV equation as a finite-dimensional dynamical system. Some results for the weakly dissipative CH equation are proved in [22] and recently, [11, 24] discussed blow-up and global existence for the weakly dissipative DP equation.

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The goal of the present paper is to study the Cauchy problem for the periodic weakly dissipative μDP equation

$$(2)\qquad \begin{cases} y_t + uy_x + 3u_x y + \lambda y &= 0, \\ y &= \mu(u) - u_{xx}, \\ u(0,x) &= u_0(x). \end{cases}$$

Here the function $u(t,x)$ is depending on time $t \geq 0$ and a space variable $x \in \mathbb{S}$ and μ is the projection $\mu(u) = \int_0^1 u(t,x)\,\mathrm{d}x$. The constant λ is assumed to be positive and the term $\lambda(\mu(u) - u_{xx})$ models energy dissipation. By the replacement $\mu(u) \mapsto u$ in (2), we obtain the weakly dissipative DP equation. Note that the quantity $E_1(u) = \int_{\mathbb{S}} y\,\mathrm{d}x$ is conserved for the DP equation and that E_1 can be interpreted as an energy, since it equals (up to a factor) a Hamiltonian function for the DP as explained in [18]. However, for the weakly dissipative DP equation, $\frac{\mathrm{d}}{\mathrm{d}t}E_1(u) = -\lambda\mu(u)$, such that $\mu(u_0) > 0$ implies that the wave's energy decreases as t increases. The weak dissipation also breaks other conservation laws of the DP equation like $E_2(u) = \int_{\mathbb{S}} yv\,\mathrm{d}x$ or $E_3(u) = \int_{\mathbb{S}} u^3\,\mathrm{d}x$, where $v = (4-\partial_x^2)^{-1}u$; cf. [21].

The general framework in which we discuss equation (2) is based on geometric and Lie theoretic techniques as introduced in [1, 7]. Equation (2) can be regarded as an evolution equation on the group $\mathrm{Diff}^s(\mathbb{S})$ of orientation-preserving H^s diffeomorphisms of the circle $\mathbb{S}$, for $s > 3/2$: The vector field $u(t,\cdot) \in H^s(\mathbb{S})$ has a unique local flow $\varphi(t,\cdot) \in \mathrm{Diff}^s(\mathbb{S})$ such that $\varphi_t \circ \varphi^{-1} = u$, $\varphi(0) = \mathrm{id}$ and $\varphi_{tt} = -F(\varphi, \varphi_t)$ with some map F defined on $\mathrm{Diff}^s(\mathbb{S}) \times H^s(\mathbb{S})$. This equation can be handled with standard ODE methods for Banach spaces. Altogether, it will turn out that the weakly dissipative μDP equation behaves quite similarly to the μDP equation (for which $\lambda = 0$) or the weakly dissipative DP equation.

The paper is organized as follows: In Section 2, we prove local well-posedness for the initial value problem (2) with $u_0 \in H^s(\mathbb{S})$ for $s > 3/2$. In Section 3, we show that for smooth initial data with zero mean, the solution $u(t,\cdot)$ of (2) can blow up in finite time. If $\mu(u_0) \neq 0$ and $\mu(u_0) - u_{0xx}$ is non-negative or non-positive, the corresponding solution $u(t,\cdot)$ will exist globally in time.

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2. Local well-posedness

In this section, we aim to establish a local well-posedness result for the Cauchy problem (2). The proof uses some geometric arguments and is based on a reformulation of the weakly dissipative μDP as a quasi-linear evolution equation; cf. [18]. We will use the notation

$$\Lambda_\mu^2 := \mu - \partial_x^2$$

and write $y = \Lambda_\mu^2 u$, $y_0 = \Lambda_\mu^2 u_0$. It is not hard to see that Λ_μ^2 is a topological isomorphism between the Sobolev spaces $H^s(\mathbb{S})$ and $H^{s-2}(\mathbb{S})$, $s \geq 2$, cf. Sect. 4; the inverse of Λ_μ^2 is denoted by Λ_μ^{-2}. In the following, we are only interested in Sobolev functions of class $s = 3$, but our first theorem deals with the more general case $s > 3/2$. Note that $H^s(\mathbb{S}) \subset C^k(\mathbb{S})$, $s > k + 1/2$, and that the square of $\|\cdot\|_s = \|\cdot\|_{H^s(\mathbb{S})}$ is the quadratic form (with respect to the L_2 inner product) induced by the operator $Q^{2s} = (1-\partial_x^2)^s$, which also defines an isomorphism between the spaces

$H^k(\mathbb{S})$ and $H^{k-2s}(\mathbb{S})$. The group of orientation-preserving circle diffeomorphisms $\mathbb{S} \to \mathbb{S}$ of class H^s is denoted by $\mathrm{Diff}^s(\mathbb{S})$, i.e.,

$$\mathrm{Diff}^s(\mathbb{S}) := \left\{ \varphi \in H^s(\mathbb{S});\ \varphi \text{ is bijective, orientation-preserving and } \varphi^{-1} \in H^s(\mathbb{S}) \right\}.$$

Observe that $T_\varphi \mathrm{Diff}^s(\mathbb{S}) \simeq H^s(\mathbb{S})$ for any $\varphi \in \mathrm{Diff}^s(\mathbb{S})$. The following lemma establishes that $\mathrm{Diff}^s(\mathbb{S})$ is a topological group for $s > 3/2$. The reader can find a proof in [19].

Lemma 1. *For $s > 3/2$, the composition map $\varphi \mapsto \omega \circ \varphi$ with an H^s function ω and the inversion map $\varphi \mapsto \varphi^{-1}$ are continuous maps $\mathrm{Diff}^s(\mathbb{S}) \to H^s(\mathbb{S})$ and $\mathrm{Diff}^s(\mathbb{S}) \to \mathrm{Diff}^s(\mathbb{S})$ respectively and*

$$\|\omega \circ \varphi\|_{H^s} \leq C(1 + \|\varphi\|_{H^s}^s)\, \|\omega\|_{H^s}\,;$$

C only depending on $\sup_{x\in\mathbb{S}} |\varphi_x(x)|$ and $\inf_{x\in\mathbb{S}} |\varphi_x(x)|$.

Before we proceed, we state the following lemma which has an interesting geometric interpretation and is derived directly from the local existence and uniqueness theorem for differential equations in Banach spaces; cf. [16].

Lemma 2. *Let $u(t,x)$ be a time-dependent H^s function on the circle for $s > 3/2$. Then the problem*

$$\begin{cases} \varphi_t(t,x) &= u(t, \varphi(t,x)), \\ \varphi(0,x) &= x, \end{cases}$$

for $x \in \mathbb{S}$ and $t \geq 0$, has a unique solution $\varphi \in C^1([0, T_{\max}), \mathrm{Diff}^s(\mathbb{S}))$, where $T_{\max} > 0$ is maximal.

Remark 3. Note that the geometric interpretation of this lemma is that $u(t,\cdot)$ can be regarded as a vector field on the sphere $\mathbb{S}$ for which we have a local flow $\varphi \in \mathrm{Diff}^s(\mathbb{S})$. Local flows have approved to be powerful tools in the analysis of model equations for 1D water waves, see [1, 7, 9, 17].

In many texts, local well-posedness results for Cauchy problems similar to (2) are obtained by applying Kato's theory for abstract quasi-linear evolution equations. We now present a method of proof which is based on a geometric argument, most importantly using local flows as introduced in the above lemma. A technical disadvantage of this method is that it does not yield a priori a maximal existence time for our solution which we will obtain inductively. The key idea is to rewrite the weakly dissipative μDP equation in the form

$$u_t + uu_x + 3\mu(u)\partial_x \Lambda_\mu^{-2} u + \lambda u = 0; \tag{3}$$

this equation is suitable for a reformulation of (2) in the geometric picture, i.e., in terms of a local flow on the group $\mathrm{Diff}^s(\mathbb{S})$. For the following well-posedness proof, our next lemma will play a key role. The explicit calculations already occur in the proof of Theorem 5.1 in [18].

Lemma 4. *Let R_φ denote the right translation map on $\mathrm{Diff}^s(\mathbb{S})$ and let $\Lambda_{\mu,\varphi}^{-2} = R_\varphi \circ \Lambda_\mu^{-2} \circ R_{\varphi^{-1}}$ and $\partial_{x,\varphi} = R_\varphi \circ \partial_x \circ R_{\varphi^{-1}}$. Then,*

$$3\mu(\xi \circ \varphi^{-1}) \left(\Lambda_\mu^{-2} \partial_x \left(\xi \circ \varphi^{-1}\right)\right) \circ \varphi = \Lambda_{\mu,\varphi}^{-2} \partial_{x,\varphi} h(\varphi, \xi) \tag{4}$$

for $h(\varphi,\xi) = 3\xi \int_{\mathbb{S}} \xi \circ \varphi^{-1}\,\mathrm{d}x$. *Furthermore, we have the identities*

$$\partial_\varphi \Lambda^{-2}_{\mu,\varphi}(v) = -\Lambda^{-2}_{\mu,\varphi}\left[(v\circ\varphi^{-1})\partial_x, \Lambda^2_\mu\right]_\varphi \Lambda^{-2}_{\mu,\varphi}, \tag{5}$$

$$\partial_\varphi \partial_{x,\varphi}(v) = \left[(v\circ\varphi^{-1})\partial_x, \partial_x\right]_\varphi, \tag{6}$$

$$\partial_\varphi h(\varphi,\xi)(v) = 3\xi \int_{\mathbb{S}} \xi\circ\varphi^{-1}\partial_x(v\circ\varphi^{-1})\,\mathrm{d}x. \tag{7}$$

Our main theorem in this section reads as follows.

Theorem 5. *Let* $s > 3/2$ *and* $u_0 \in H^s(\mathbb{S})$. *Then there is a maximal time* $T \in (0,\infty]$ *and a unique solution*

$$u \in C\left([0,T); H^s(\mathbb{S})\right) \cap C^1\left([0,T); H^{s-1}(\mathbb{S})\right)$$

of the Cauchy problem (2) *which depends continuously on the initial data* u_0, *i.e., the mapping*

$$H^s(\mathbb{S}) \to C\left([0,T); H^s(\mathbb{S})\right) \cap C^1\left([0,T); H^{s-1}(\mathbb{S})\right), \quad u_0 \mapsto u(\cdot, u_0)$$

is continuous.

Proof. Writing

$$Au = 3\mu(u)\partial_x\Lambda^{-2}_\mu u + \lambda u = A_0 u + \lambda u,$$

Eq. (3) shows that (2) is equivalent to $u_t + uu_x = -Au$. Let $\varphi \in \mathrm{Diff}^s(\mathbb{S})$ denote the local flow for the vector field $u(t,\cdot)$ according to Lemma 2, i.e., $\varphi(t)$ is defined on some maximal interval $[0, T_{\max})$ and is C^1, with $u\circ\varphi = \varphi_t$ and $\varphi(0) = \mathrm{id}$. If we can differentiate φ_t once again, we may derive the identity

$$\varphi_{tt} = (u_t + uu_x)\circ\varphi = -A(\varphi_t\circ\varphi^{-1})\circ\varphi.$$

Let $F(\varphi,\varphi_t) := R_\varphi \circ A \circ R_{\varphi^{-1}}\varphi_t$ so that

$$\varphi_{tt} = -F(\varphi,\varphi_t), \quad \varphi_t(0) = u_0, \quad \varphi(0) = \mathrm{id}, \tag{8}$$

which is an ordinary second order initial value problem. Interestingly, any solution φ to the initial value problem (8) yields a solution $u = \varphi_t\circ\varphi^{-1}$ to the weakly dissipative μDP equation in its initial form, with the desired regularity properties. This motivates to study the Cauchy problem for the periodic weakly dissipative μDP in the reformulation (8) on the diffeomorphism group of the circle.

We now decompose $F = F_1 + F_2$ with $F_1 = R_\varphi\circ A_0\circ R_{\varphi^{-1}}$ and F_2 just being multiplication with λ. Both, F_1 and F_2 are Fréchet differentiable in a neighborhood of any $(\varphi,\xi) \in T\mathrm{Diff}^s(\mathbb{S}) \simeq \mathrm{Diff}^s(\mathbb{S}) \times H^s(\mathbb{S})$ and the directional derivatives $\partial_\varphi F_i$, $\partial_\xi F_i$, $i = 1, 2$, are bounded linear operators on $H^s(\mathbb{S})$ with continuous dependence on (φ,ξ); this is trivial for F_2 and proved in [18] for F_1 by applying Lemma 4.

Since F is continuously differentiable near $(\mathrm{id}, 0)$, the standard local existence theorem for Banach spaces (cf. [16]) establishes the local well-posedness of (8), i.e., there is a time $T_1 > 0$ and a unique solution (φ,φ_t) of (8) on $[0, T_1]$ with continuous dependence on t and u_0. To show that there is a maximal interval of existence, we apply the Cauchy-Lipschitz Theorem once more to the problem $\varphi_{tt} = -F(\varphi,\varphi_t)$ with initial data $(\varphi(T_1), \varphi_t(T_1))$ to continue the solution (φ,φ_t) to a solution on a time interval $[0, T_2]$ with $T_1 < T_2$. Iterating this procedure, we obtain a monotonically increasing sequence $(T_n)_{n\in\mathbb{N}}$ and an associated sequence of solutions

$$\varphi_n \in C^2\left([0,T_n]; \mathrm{Diff}^s(\mathbb{S})\right), \quad \varphi_{n+1}|_{[0,T_n]} = \varphi_n.$$

If $(T_n)_{n\in\mathbb{N}}$ is bounded, $T_n \to T$ as $n \to \infty$, with a real number T; otherwise, $T_n \to \infty$.

Now the well-posedness of the problem (2) is a simple consequence of the relations $u = \varphi_t \circ \varphi^{-1}$ and $u_t = -uu_x - Au$ and the fact that $\mathrm{Diff}^s(\mathbb{S})$ is a topological group whenever $s > 3/2$. □

Remark 6. As explained in the proof of Theorem 5, let $(T_n)_{n\in\mathbb{N}}$ denote the strictly increasing sequence describing the continuation of our solution $u(t,x)$ in $H^s(\mathbb{S})$. If $(T_n)_{n\in\mathbb{N}}$ is bounded, we say that the solution u has a *finite* existence time, where $T_n \to \infty$ means that the solution exists *globally* in time. It is an interesting problem and the aim of the following sections to describe the behavior of finite-time solutions as $t \to T$ from below and to find criteria for the global existence of strong solutions as well as so-called *finite-time blow-up*.

3. Global well-posedness and blow-up

In physics, a breaking wave is a wave whose amplitude reaches a critical level at which some process can suddenly start to occur that causes large amounts of wave energy to be transformed in turbulent kinetic energy. At this point, simple physical models describing the dynamics of the wave will often become invalid, particularly those which assume linear behavior. Wave breaking has been studied for various classes of non-linear model equations for 1D water waves and a reasonable way is to show that there is a finite-time solution u satisfying an L_∞-bound for all $t \in [0,T)$ so that the norm of u is unbounded as $t \to T$ if and only if the first order derivative u_x approaches $-\infty$ as $t \to T$ from below (cf., e.g., [11] for a discussion of the DP equation with a dissipative term). The physical interpretation then is that the wave steepens, while the height of its crests stays bounded, until wave breaking occurs in the sense that u ceases to be a classical solution.

In this section, we first describe the blow-up of finite-time solutions of (2) in terms of the first order derivative and then discuss precise blow-up settings. Here and in what follows, we will restrict ourselves to $s = 3$. Recall that $H^3(\mathbb{S})$-functions are of class C^2 such that there will be no boundary terms when performing integration by parts.

Theorem 7. *Given $u_0 \in H^3(\mathbb{S})$, the solution u of* (2) *obtained in Theorem* 5 *blows up in finite time $T > 0$ if and only if*

$$\liminf_{t\to T} \min_{x\in\mathbb{S}} u_x(t,x) = -\infty.$$

Proof. Let $T > 0$ be the maximal time of existence of the solution u to Eq. (2) with initial data u_0. Since $H^3(\mathbb{S}) \subset C^2(\mathbb{S})$ we find that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}} y^2\,\mathrm{d}x &= 2\int_{\mathbb{S}} yy_t\,\mathrm{d}x \\
&= -2\int_{\mathbb{S}} uy_xy\,\mathrm{d}x - 6\int_{\mathbb{S}} u_xy^2\,\mathrm{d}x - 2\lambda\int_{\mathbb{S}} y^2\,\mathrm{d}x \\
&= -5\int_{\mathbb{S}} u_xy^2\,\mathrm{d}x - 2\lambda\int_{\mathbb{S}} y^2\,\mathrm{d}x. \qquad (9)
\end{aligned}$$

If we assume $u_0 \in H^4(\mathbb{S})$ and use that $H^4(\mathbb{S}) \subset C^3(\mathbb{S})$, we can obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}} y_x^2\,\mathrm{d}x &= 2\int_{\mathbb{S}} y_x y_{tx}\,\mathrm{d}x \\
&= -2\int_{\mathbb{S}} y_x y_{xx} u\,\mathrm{d}x - 8\int_{\mathbb{S}} y_x^2 u_x\,\mathrm{d}x - 6\int_{\mathbb{S}} y y_x u_{xx}\,\mathrm{d}x - 2\lambda\int_{\mathbb{S}} y_x^2\,\mathrm{d}x \\
&= -7\int_{\mathbb{S}} y_x^2 u_x\,\mathrm{d}x - 2\lambda\int_{\mathbb{S}} y_x^2\,\mathrm{d}x. \qquad (10)
\end{aligned}
$$

Adding (9) and (10) we get

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|y\|_{H^1}^2 = -7\int_{\mathbb{S}} y_x^2 u_x\,\mathrm{d}x - 5\int_{\mathbb{S}} u_x y^2\,\mathrm{d}x - 2\lambda\|y\|_{H^1}^2. \qquad (11)
$$

Next we observe that (11) also holds true for $u_0 \in H^3(\mathbb{S})$: We approximate u_0 in $H^3(\mathbb{S})$ by functions $u_0^n \in H^4(\mathbb{S})$, $n \geq 1$. Let $u^n = u^n(\cdot, u_0^n)$ be the solution of (2) with initial data u_0^n. By Theorem 5 we know that

$$
u^n \in C\left([0,T_n); H^4(\mathbb{S})\right) \cap C^1\left([0,T_n); H^3(\mathbb{S})\right), \quad n \geq 1,
$$

$$
y^n = \mu(u^n) - u_{xx}^n \in C\left([0,T_n); H^2(\mathbb{S})\right) \cap C^1\left([0,T_n); H^1(\mathbb{S})\right), \quad n \geq 1,
$$

$u^n \to u$ in $H^3(\mathbb{S})$ and $T_n \to T$ as $n \to \infty$. Since $u_0^n \in H^4(\mathbb{S})$, we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}} (y_x^n)^2\,\mathrm{d}x = -7\int_{\mathbb{S}} (y_x^n)^2 u_x^n\,\mathrm{d}x - 2\lambda\int_{\mathbb{S}} (y_x^n)^2\,\mathrm{d}x.
$$

Since $u_n \to u$ in $H^3(\mathbb{S})$ it follows that $u_x^n \to u_x$ in $L_\infty(\mathbb{S})$ as $n \to \infty$. Note also that $y^n \to y$ in $H^1(\mathbb{S})$ and $y_x^n \to y_x$ in $L_2(\mathbb{S})$ as $n \to \infty$. We deduce that, as $n \to \infty$, (10) also holds for $u_0 \in H^3(\mathbb{S})$. If u_x is bounded from below on $[0,T)$, i.e., $u_x \geq -c$, where c is a positive constant, then we can apply Gronwall's inequality to (11) and have

$$
\|y\|_{H^1}^2 \leq \|y_0\|_{H^1}^2 \exp((7c - 2\lambda)t).
$$

This shows that $\|u\|_{H^3}$ does not blow up in finite time. The converse direction follows from Sobolev's embedding theorem. This completes the proof of our assertion. $\square$

Remark 8. The above proof shows that if u_x stays bounded, then u also persists in H^3. Thus Theorem 7 provides us with a sufficient criterion for global existence, namely the boundedness of $\|u_x(t,\cdot)\|_\infty$ as t approaches T from below.

It is well known that the mean $\mu(u)$ of a solution $u(t,\cdot)$ of the μDP equation is conserved, i.e., $\mu(u_0) = \mu(u)$; see [8]. We now show that the mean $\mu(u)$ of a solution of the weakly dissipative μDP equation decreases exponentially as t increases from zero. More precisely, we prove that the damping constant is equal to the dissipation parameter λ.

Lemma 9. *Let $u_0 \in H^3(\mathbb{S})$ and denote by $u(t,\cdot)$ the solution of (2) obtained in Theorem 5. Then the mean of u satisfies*

$$
\mu(u) = \mu(u_0)e^{-\lambda t}
$$

for $t \geq 0$ in the existence interval of u. In particular, if $\mu(u_0) = 0$, then the mean of the solution u is conserved.

Proof. By differentiating under the integral sign and using (3), we obtain that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mu(u) &= \mu\left(-uu_x - 3\mu(u)\partial_x\Lambda_\mu^{-2}u - \lambda u\right)\\ &= -\mu\left(\frac{1}{2}\partial_x\left(u^2\right)\right) - 3\mu(u)\mu\left(\partial_x\Lambda_\mu^{-2}u\right) - \lambda\mu(u),\end{aligned} \tag{12}$$

as long as the solution $u(t,\cdot) \in H^3(\mathbb{S})$ exists. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu(u) = -\lambda\mu(u)$$

from which the lemma follows. □

With the help of Lemma 9, we are able to establish the following blow-up setting. It is important to notice that our result shows the blow-up of smooth initial data. A corresponding result for μDP (the case $\lambda = 0$) can be found in [18].

Theorem 10. *Assume that $0 \neq u_0 \in C^\infty(\mathbb{S})$ has zero mean and that there is $x^* \in \mathbb{S}$ satisfying*

$$0 < 1 + \frac{\lambda}{u_{0x}(x^*)} < 1. \tag{13}$$

Let u be the corresponding solution of (2). Then there is $0 < \tau < \infty$ such that $\|u_x(t)\|_\infty$ blows up as $t \to \tau$. In particular, the solution u blows up in the H^3-norm in finite time.

Proof. Differentiating equation (3) with respect to x and the identity $\partial_x^2\Lambda_\mu^{-2} = \mu - 1$ (see Sect. 4) yield

$$u_{tx} + uu_{xx} + u_x^2 + \lambda u_x = 3\mu(u)(u - \mu(u)).$$

By Lemma 9, it follows that the right hand side equals zero. Again, we denote by φ the local flow of the time-dependent vector field $u(t,\cdot)$, i.e., $\varphi_t = u \circ \varphi$. We set

$$w := \frac{\varphi_{tx}}{\varphi_x} = u_x \circ \varphi$$

and with

$$\varphi_{ttx} = \left[(u_{tx} + uu_{xx} + u_x^2) \circ \varphi\right]\varphi_x$$

we obtain

$$w_t = \frac{\varphi_{ttx}\varphi_x - (\varphi_{tx})^2}{\varphi_x^2} = (u_{tx} + uu_{xx}) \circ \varphi$$

and hence

$$w_t + w^2 + \lambda w = 0.$$

With $\Gamma := -\lambda < 0$, we finally arrive at the logistic equation

$$w_t = w(\Gamma - w)$$

and standard ODE techniques show that the solution is given by

$$w(t) = \frac{\Gamma}{1 + \left(\frac{\Gamma}{w(0)} - 1\right)e^{-\Gamma t}}.$$

Recall that $w(0) = u_{0x}(x)$. By our assumption on u_0, we can find a point $x^* \in \mathbb{S}$ with

$$0 < 1 + \frac{\lambda}{u_{0x}(x^*)} < 1.$$

If we set

$$\tau = -\frac{1}{\lambda} \ln \left(1 + \frac{\lambda}{u_{0x}(x^*)} \right),$$

it follows that the solution must blow up in the H^3-norm. □

Remark 11. Condition (13) means that we can find $x^* \in \mathbb{S}$ such that

(*i*) $u_{0x}(x^*) < 0$ and

(*ii*) $|u_{0x}(x^*)| > \lambda$.

Since we assume $\mu(u_0) = 0$, it follows that u_0 must change sign. Since $u_0 \in C^\infty(\mathbb{S})$, u_0 has to change sign at least twice and so it is always possible to find $x^* \in \mathbb{S}$ satisfying (*i*). Our second condition says that the slope of u_0 must decrease λ in order to obtain blow-up: The larger the dissipation given by λ, the larger must $|u_{0x}|$ be locally in order to obtain a blow-up. So (13) is a non-trivial common condition for λ and u_0 in our blow-up setting.

The following lemma is similar to Lemma 2.2. in [11]. Furthermore, we see that as $\lambda \to 0$, we obtain the conservation of the quantity $(y \circ \varphi)\varphi_x^3$, which is explained in [9, 18] for the DP and the μDP.

Lemma 12. *Let $u_0 \in H^3(\mathbb{S})$ and let $T > 0$ be the maximal existence time of the corresponding solution $u(t,x)$ according to Theorem* 5*. Let φ be the local flow of u according to Lemma 2. Then we have*

$$y(t, \varphi(t,x))\varphi_x^3(t,x) = y_0(x)e^{-\lambda t}.$$

Proof. An easy calculation shows that the function

$$[0,T) \mapsto \mathbb{R}, \quad t \mapsto e^{\lambda t} y(t, \varphi(t,x))\varphi_x^3(t,x)$$

is constant. Using $\varphi(0) = \mathrm{id}$ and $\varphi_x(0) = 1$, we are done. □

Finally, we come to the following global well-posedness result. Note that our assumptions on the initial condition u_0 are quite similar to the ones in Theorem 5.4. in [18].

Theorem 13. *Assume that $u_0 \in H^3(\mathbb{S})$ has positive mean and satisfies the condition $\Lambda_\mu^2 u_0 \geq 0$. Then the Cauchy problem* (2) *has a unique global solution in $C([0,\infty), H^3(\mathbb{S})) \cap C^1([0,\infty), H^2(\mathbb{S}))$.*

Proof. Let $u(t,\cdot) \in H^3(\mathbb{S})$, $t \in [0,T)$, denote the solution of (2) obtained in Theorem 5. According to Theorem 7, we only have to show that $\|u_x(t,\cdot)\|_\infty$ stays bounded as t approaches T from below. Note that, for any periodic function w, differentiating formula (14) with $f = \Lambda_\mu^2 w$ yields

$$\|\partial_x w\|_\infty \leq C \left\| \Lambda_\mu^2 w \right\|_{L_1},$$

with a constant $C \geq 0$. Now Lemma 12 and the assumption $\Lambda_\mu^2 u_0 \geq 0$ imply that

$$\left\| \Lambda_\mu^2 u \right\|_{L_1} = \mu \left(\Lambda_\mu^2 u \right).$$

Using Lemma 9, we have the estimate

$$\|\partial_x u(t,\cdot)\|_\infty \leq C \int_0^1 \Lambda_\mu^2 u \,\mathrm{d}x = C\mu(u) \leq C\mu(u_0) < \infty,$$

from which the indefinite persistence of the solution u follows. □

It is clear that Theorem 13 also holds if $\Lambda_\mu^2 u_0 \leq 0$ and $\mu(u_0) < 0$.

4. Appendix

We denote by $H^k = H^k(\mathbb{S})$, $k \geq 0$, the Sobolev space of periodic functions. If $k \in \mathbb{N}_0$, H^k is the space of all $L_2(\mathbb{S})$-functions f with square integrable distributional derivatives up to the order k, $\partial_x^j f \in L_2(\mathbb{S})$, $j = 0, \ldots, k$. Endowed with the norm

$$\|f\|_k^2 = \sum_{j=0}^k \int_{\mathbb{S}} (\partial_x^j f)^2(x)\,\mathrm{d}x = \sum_{i=0}^k \left\langle \partial_x^j f, \partial_x^j f \right\rangle_{L_2(\mathbb{S})} = \sum_{j=0}^k \left\|\partial_x^j f\right\|_{L_2(\mathbb{S})}^2,$$

the spaces H^k become Hilbert spaces. Note that we have $H^0 = L_2(\mathbb{S})$. To define the spaces H^k for general $k \geq 0$, we make use of the fact that the Fourier transform $\mathscr{F}$ maps any square integrable function f on $\mathbb{S}$ to its Fourier series $(\hat{f}(n))_{n\in\mathbb{Z}}$ so that $f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{2\pi i n x}$. The space H^k consists of all $f \in L_2(\mathbb{S})$ with the property that the quadratic form $\left\langle Q^{2k} f, f\right\rangle_{L_2(\mathbb{S})}$ has a finite value, where $Q = (1-\partial_x^2)^{1/2}$ is the elliptic pseudo-differential operator with the symbol $(1+4\pi^2 n^2)^{1/2}$, i.e.,

$$(\mathscr{F}(Q^k f))(n) = (1+4\pi^2 n^2)^{k/2} \hat{f}(n).$$

We thus have

$$H^k(\mathbb{S}) := \left\{ f \in L_2(\mathbb{S});\ \|f\|_k^2 = \sum_{n\in\mathbb{Z}} \left|(\mathscr{F}(Q^k f))(n)\right|^2 < \infty \right\}.$$

It is easy to check that the operator $\Lambda_\mu^2 = \mu - \partial_x^2$ has the inverse

$$\begin{aligned}(\Lambda_\mu^{-2} f)(x) &= \left(\frac{1}{2}x^2 - \frac{1}{2}x + \frac{13}{12}\right)\int_0^1 f(a)\,\mathrm{d}a + \left(x - \frac{1}{2}\right)\int_0^1\int_0^a f(b)\,\mathrm{d}b\,\mathrm{d}a \\ &\quad - \int_0^x \int_0^a f(b)\,\mathrm{d}b\,\mathrm{d}a + \int_0^1\int_0^a\int_0^b f(c)\,\mathrm{d}c\,\mathrm{d}b\,\mathrm{d}a. \qquad (14)\end{aligned}$$

To obtain Green's function $g(x-x')$ for Λ_μ^{-2}, we observe that applying Λ_μ^2 to

$$g(x) = \frac{1}{2}x^2 - \frac{1}{2}|x| + \frac{13}{12}$$

gives the delta distribution. Hence

$$(\Lambda_\mu^{-2} f)(x) = \int_0^1 g(x-x') f(x')\,\mathrm{d}x'.$$

Moreover, we see that $[\partial_x, \Lambda_\mu^{-2}] = 0$ and $\partial_x^2 \Lambda_\mu^{-2} = \mu - 1$. It is also easy to verify that $\Lambda_\mu^2 \colon H^k \to H^{k-2}$, $k \geq 2$, is a topological isomorphism: For any $f \in H^k$ we have

$$\left\|\Lambda_\mu^2 f\right\|_{k-2}^2 = \left\| \hat{f}(0) + \sum_{n\neq 0} \hat{f}(n) 4\pi^2 n^2 e^{2\pi i n x} \right\|_{k-2}^2 \leq 2 \sum_{n\in\mathbb{Z}} (1+4\pi^2 n^2)^k \left|\hat{f}(n)\right|^2 = 2\,\|f\|_k^2\,;$$

together with (14) the open mapping theorem achieves the desired result.

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Institute for Applied Mathematics, University of Hannover, D-30167 Hannover, Germany

E-mail address: kohlmann@ifam.uni-hannover.de